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Notes on non-archimedean topological groups

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Dedicated to Professor Dikran Dikranjan on his 60th birthday

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ABSTRACT

We show that the Heisenberg type group $H_X = (\mathbb{Z}_2 \oplus V) \leftthreetimes V^*$, with the discrete Boolean group $V := C(X, \mathbb{Z}_2)$, canonically defined by any Stone space X, is always minimal. That is, H_X does not admit any strictly coarser Hausdorff group topology. This leads us to the following result: for every (locally compact) non-archimedean G there exists a (resp., locally compact) non-archimedean minimal group M such that G is a group retract of M. For discrete groups G the latter was proved by S. Dierolf and U. Schwanengel (1979) [6]. We unify some old and new characterization results for non-archimedean groups.

1. Introduction and preliminaries

A topological group is *non-archimedean* if it has a local base at the identity consisting of open subgroups. This class of groups coincides with the class of topological subgroups of the homeomorphism groups Homeo(X), where X runs over *Stone spaces* (= compact zero-dimensional spaces) and Homeo(X) carries the usual compact open topology. Recall that by Stone's representation theorem, there is a duality between the category of Stone spaces and the category of Boolean algebras. The class $\mathcal{NA}$ of non-archimedean groups and the related class of ultra-metric spaces have many applications. For instance, in non-archimedean functional analysis, in descriptive set theory, computer science, etc. See, e.g., [36,3,22,21,43] and references therein.

In the present paper we provide some applications of generalized Heisenberg groups, with emphasis on minimality properties, in the theory of $\mathcal{NA}$ groups and actions on Stone spaces.

Recall that a Hausdorff topological group G is *minimal* (Stephenson [38] and Doïchinov [12]) if it does not admit a strictly coarser Hausdorff group topology, or equivalently, if every injective continuous group homomorphism $G \to P$ into a Hausdorff topological group is a topological group embedding.

If otherwise is not stated all topological groups and spaces in this paper are assumed to be Hausdorff. We say that an additive topological group $(G, +)$ is a *Boolean group* if $x + x = 0$ for every $x \in G$. As usual, a G*-space* X is a topological space X with a continuous group action $\pi : G \times X \to X$ of a topological group G. We say that X is a G*-group* if, in addition, X is a topological group and all g-translations, $\pi^g : X \to X,\ x \mapsto gx := \pi(g, x)$, are automorphisms of X. For every G-group X we denote by $X \leftthreetimes G$ the corresponding topological semidirect product.

To every Stone space X we associate a (locally compact 2-step nilpotent) Heisenberg type group

$$H_X = (\mathbb{Z}_2 \oplus V) \leftthreetimes V^*,$$

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where $V := C(X, \mathbb{Z}_2)$ is a discrete Boolean group which can be identified with the group of all clopen subsets of X (symmetric difference is the group operation). $V^* := \mathrm{Hom}(V, \mathbb{Z}_2)$ is the compact group of all group homomorphisms into the two element cyclic group $\mathbb{Z}_2$. V^* acts on $\mathbb{Z}_2 \oplus V$ in the following way: every $(f, (a, x)) \in V^* \times (\mathbb{Z}_2 \oplus V)$ is mapped to $(a + f(x), x) \in \mathbb{Z}_2 \oplus V$. The group operation on H_X is defined as follows: for

$$u_1 = (a_1, x_1, f_1), \qquad u_2 = (a_2, x_2, f_2) \in H_X$$

we define

$$u_1 u_2 = \big(a_1 + a_2 + f_1(x_2), x_1 + x_2, f_1 + f_2\big).$$

In Section 4 we study some properties of H_X and show in particular (Theorem 4.1) that the (locally compact) Heisenberg group $H_X = (\mathbb{Z}_2 \times V) \rtimes V^*$ is minimal and non-archimedean for every Stone space X.

Every Stone space X is naturally embedded into $V^* := \mathrm{Hom}(V, \mathbb{Z}_2)$ by the natural map $\delta : X \to V^*$, $x \mapsto \delta_x$ where $\delta_x(f) := f(x)$. Every δ_x can be treated as a 2-valued measure on X. Identifying X with $\delta(X) \subset V^*$ we get a restricted evaluation map $V \times X \to \mathbb{Z}_2$ which in fact is the evaluation map of the Stone duality. Note that the role of $\delta : X \to V^*$ for a compact space X is similar to the role of the Gelfand map $X \to C(X)^*$, representing X via the point measures.

For every action of a group $G \subset \mathrm{Homeo}(X)$ on a Stone space X we can deal with a G-space version of the classical Stone duality. The map $\delta : X \to V^*$ is a G-map of G-spaces. Furthermore, a deeper analysis shows (Theorem 4.4) that every topological subgroup $G \subset \mathrm{Homeo}(X)$ induces a continuous action of G on H_X by automorphisms such that the corresponding semidirect product $H_X \rtimes G$ is a minimal group.

We then conclude (Corollary 4.5) that every (locally compact) non-archimedean group is a group retract of a (resp., locally compact) minimal non-archimedean group. It covers a result of Dierolf and Schwanengel [6] (see also Example 3.5 below) which asserts that every discrete group is a group retract of a locally compact non-archimedean minimal group.

Section 2 contains additional motivating results and questions. Several interesting applications of generalized Heisenberg groups can be found in the papers [25–27,11,28,8,9,37].

Studying the properties of the Heisenberg group H_X, we get a unified approach to several (mostly known) equivalent characterizations of the class $\mathcal{NA}$ of non-archimedean groups (Lemma 3.2 and Theorem 5.1). In particular, we show that the class of all topological subgroups of $\mathrm{Aut}(K)$, for compact abelian groups K, is precisely $\mathcal{NA}$.

2. Minimality and group representations

Clearly, every compact topological group is minimal. Trivial examples of nonminimal groups are: the group $\mathbb{Z}$ of all integers (or any discrete infinite abelian group) and $\mathbb{R}$, the topological group of all reals. By a fundamental theorem of Prodanov and Stoyanov [32] every abelian minimal group is precompact. For more information about minimal groups see review papers of Dikranjan [7] and Comfort, Hofmann and Remus [5], a book of Dikranjan, Prodanov and Stoyanov [10] and a recent book of Lukacs [23].

Unexpectedly enough many non-compact naturally defined topological groups are minimal.

Remark 2.1. Recall some nontrivial examples of minimal groups.

(1) Prodanov [31] showed that the p-adic topologies are the only precompact minimal group topologies on $\mathbb{Z}$.
(2) Symmetric topological groups S_X (Gaughan [15]).
(3) $\mathrm{Homeo}(\{0,1\}^{\aleph_0})$ (see Gamarnik [14] and also Uspenskij [42] for a more general case).
(4) $\mathrm{Homeo}[0,1]$ (Gamarnik [14]).
(5) The semidirect product $\mathbb{R} \rtimes \mathbb{R}_+$ (Dierolf and Schwanengel [6]). More general cases of minimal (so-called *admissible*) semidirect products were studied by Remus and Stoyanov [35]. By [26], $\mathbb{R}^n \rtimes \mathbb{R}_+$ is minimal for every $n \in \mathbb{N}$.
(6) Every connected semisimple Lie group with finite center, e.g., $SL_n(\mathbb{R})$, $n \geqslant 2$ (Remus and Stoyanov [35]).
(7) The full unitary group $U(H)$ (Stoyanov [39]).

One of the immediate difficulties is the fact that minimality is not preserved by quotients and (closed) subgroups. See for example item (5) with minimal $\mathbb{R} \rtimes \mathbb{R}_+$ where its canonical quotient $\mathbb{R}_+$ (the positive reals) and the closed normal subgroup $\mathbb{R}$ are nonminimal. As a contrast note that in a minimal *abelian* group every closed subgroup is minimal [10].

In 1983 Pestov raised the conjecture that every topological group is a group retract of a minimal group. Note that if $f : M \to G$ is a group retraction then necessarily G is a quotient of M and also a closed subgroup in M. Arhangel'skiĭ asked the following closely related questions:

Question 2.2. *([2,30]) Is every topological group a quotient of a minimal group? Is every topological group a closed subgroup of a minimal group?*

By a result of Uspenskij [41] every topological group is a subgroup of a minimal group M which is Raikov-complete, topologically simple and Roelcke-precompact.

Recently a positive answer to Pestov's conjecture (and hence to Question 2.2 of Arhangel'skiĭ) was obtained in [28]. The proof is based on methods (from [25]) of constructing minimal groups using group representations on Banach spaces and involving generalized Heisenberg groups.

According to [25] every locally compact *abelian* group is a group retract of a minimal locally compact group. It is an open question whether the same is true in the non-abelian case.

Question 2.3. *([25,28,5]) Is it true that every locally compact group G is a group retract* (*at least a subgroup or a quotient*) *of a locally compact minimal group?*

A more general natural question is the following:

Question 2.4. *([25]) Let $\mathcal{K}$ be a certain class of topological groups and* $\underline{\min}$ *denotes the class of all minimal groups. Is it true that every $G \in \mathcal{K}$ is a group retract of a group $M \in \mathcal{K} \cap \underline{\min}$?*

So Corollary 4.5 gives a partial answer to Questions 2.3 and 2.4 in the class $\mathcal{K} := \mathcal{NA}$ of non-archimedean groups.

Remark 2.5. Note that by [28, Theorem 7.2] we can present any topological group G as a group retraction $M \to G$, where M is a minimal group having the same weight and character as G. Furthermore, if G is Raikov-complete then M also has the same property. These results provide in particular a positive answer to Question 2.4 in the following basic classes: second countable groups, metrizable groups, Polish groups.

2.1. Minimality properties of actions

Definition 2.6. Let $\alpha : G \times X \to X$, $\alpha(g,x) = gx$ be a continuous action of a Hausdorff topological group (G,σ) on a Hausdorff topological space (X,τ). The action α is said to be:

(1) *Algebraically exact* if $\ker_\alpha := \{g \in G\colon\ gx = x\ \forall x \in X\}$ is the trivial subgroup $\{e\}$.
(2) *Topologically exact* (*t-exact*, in short) if there is no strictly coarser, not necessarily Hausdorff, group topology $\sigma' \subsetneq \sigma$ on G such that α is (σ',τ,τ)-continuous.

Remark 2.7.

(1) Every topologically exact action is algebraically exact. Indeed, otherwise $\ker_\alpha$ is a nontrivial subgroup in G. Then the preimage group topology $\sigma' \subset \sigma$ on G induced by the onto homomorphism $G \to G/\ker_\alpha$ is not Hausdorff (in particular, it differs σ) and the action remains (σ',τ,τ)-continuous.
(2) On the other hand, if α is algebraically exact then it is topologically exact if and only if for every strictly coarser *Hausdorff* group topology $\sigma' \subsetneq \sigma$ on G the action α is not (σ',τ,τ)-continuous. Indeed, since α is algebraically exact and (X,τ) is Hausdorff then every coarser group topology σ' on G which makes the action (σ',τ,τ)-continuous must be Hausdorff.

Let X be a locally compact group and $\mathrm{Aut}(X)$ be the group of all automorphisms endowed with the *Birkhoff topology* (see [16, §26] and [10, p. 260]). Some authors use the name *Braconnier topology* (see [4]).

The latter is a group topology on $\mathrm{Aut}(X)$ and has a local base formed by the sets

$$\mathcal{B}(K,O) := \left\{ f \in \mathrm{Aut}(X)\colon\ f(x) \in Ox \text{ and } f^{-1}(x) \in Ox\ \forall x \in K \right\}$$

where K runs over compact subsets and O runs over neighborhoods of the identity in X. In the sequel $\mathrm{Aut}(X)$ is always equipped with the Birkhoff topology. It equals to the Arens *g-topology* [1,4]. If X is compact then the Birkhoff topology coincides with the usual compact-open topology. If X is discrete then the Birkhoff topology on $\mathrm{Aut}(X) \subset X^X$ coincides with the pointwise topology.

Lemma 2.8. *In each of the following cases the action of G on X is t-exact*:

(1) *([25]) Let X be a locally compact group and G be a subgroup of* $\mathrm{Aut}(X)$.
(2) *Let G be a topological subgroup of* $\mathrm{Homeo}(X)$, *the group of all autohomeomorphisms of a compact space X with the compact open topology.*
(3) *Let G be a subgroup of $Is(X,d)$ the group of all isometries of a metric space (X,d) with the pointwise topology.*

Proof. Straightforward. □

Remark 2.9. Every locally compact abelian group G can be embedded into the group $\mathrm{Aut}(X)$, where X is a locally compact abelian group (with $X := \mathbb{T} \times G^*$, [25, Prop. 2.3]). Note that for locally compact subgroups of $\mathrm{Aut}(X)$ [25, Theorem 4.4] positively resolves Question 2.3. See also Remark 5.2(3).

2.2. From minimal dualities to minimal groups

In this subsection we recall some definitions and results from [25,28].

Let E, F, A be abelian additive topological groups. A map $w : E \times F \to A$ is said to be *biadditive* if the induced mappings

$$w_x : F \to A, \qquad w_f : E \to A, \qquad w_x(f) := w(x, f) =: w_f(x)$$

are homomorphisms for all $x \in E$ and $f \in F$.

A biadditive mapping $w : E \times F \to A$ is *separated* if for every pair (x_0, f_0) of nonzero elements there exists a pair (x, f) such that $f(x_0) \neq 0_A$ and $f_0(x) \neq 0_A$.

A continuous separated biadditive mapping $w : (E, \sigma) \times (F, \tau) \to A$ is *minimal* if for every coarser pair (σ_1, τ_1) of Hausdorff group topologies $\sigma_1 \subseteq \sigma,\ \tau_1 \subseteq \tau$ such that $w : (E, \sigma_1) \times (F, \tau_1) \to A$ is continuous, it follows that $\sigma_1 = \sigma$ and $\tau_1 = \tau$.

Let $w : E \times F \to A$ be a continuous biadditive mapping. Consider the action: $w^{\triangledown} : F \times (A \oplus E) \to A \oplus E,\quad w^{\triangledown}(f, (a, x)) = (a + w(x, f), x)$. Denote by $H(w) = (A \oplus E) \leftthreetimes F$ the topological semidirect product of F and the direct sum $A \oplus E$. The group operation on $H(w)$ is defined as follows: for a pair

$$u_1 = (a_1, x_1, f_1), \qquad u_2 = (a_2, x_2, f_2)$$

we define

$$u_1 u_2 = \big(a_1 + a_2 + f_1(x_2), x_1 + x_2, f_1 + f_2\big)$$

where, $f_1(x_2) = w(x_2, f_1)$. Then $H(w)$ becomes a Hausdorff topological group which is said to be a *generalized Heisenberg group* (induced by w).

Let G be a topological group and let $w : E \times F \to A$ be a continuous biadditive mapping. A continuous *birepresentation* of G in w is a pair (α_1, α_2) of continuous actions by group automorphisms $\alpha_1 : G \times E \to E$ and $\alpha_2 : G \times F \to F$ such that w is G-invariant, i.e., $w(gx, gf) = w(x, f)$.

The birepresentation ψ is said to be *t-exact* if $\ker(\alpha_1) \cap \ker(\alpha_2) = \{e\}$ and for every strictly coarser *Hausdorff* group topology on G the birepresentation does not remain continuous. For instance, if one of the actions α_1 or α_2 is t-exact then clearly ψ is t-exact.

Let ψ be a continuous G-birepresentation

$$\psi = (w : E \times F \to A,\ \alpha_1 : G \times E \to E,\ \alpha_2 : G \times F \to F).$$

The topological semidirect product $M(\psi) := H(w) \leftthreetimes_{\pi} G$ is said to be the *induced group*, where the action $\pi : G \times H(w) \to H(w)$ is defined by

$$\pi\big(g, (a, x, f)\big) = (a, gx, gf).$$

Fact 2.10. *Let $w : E \times F \to A$ be a minimal biadditive mapping and A is a minimal group. Then*

(1) *([9, Corollary 5.2]) The Heisenberg group $H(w)$ is minimal.*
(2) *([25, Theorem 4.3] and [28]) If ψ is a t-exact G-birepresentation in w then the induced group $M(\psi)$ is minimal.*

Fact 2.11. *([25]) Let G be a locally compact abelian group and $G^* := \mathrm{Hom}(G, \mathbb{T})$ be the dual (locally compact) group. Then the canonical evaluation mapping*

$$G \times G^* \to \mathbb{T}$$

is minimal and the corresponding Heisenberg group $H = (\mathbb{T} \oplus G) \leftthreetimes G^*$ is minimal.

3. Some facts about non-archimedean groups and uniformities

3.1. Non-archimedean uniformities

For information on *uniform spaces*, we refer the reader to [13] (in terms of *entourages*) and to [19] (via *coverings*). If μ is a uniformity for X in terms of coverings, then the collection of elements of μ which are *finite* coverings of X forms a base for a topologically compatible uniformity for X which we denote by μ_{fin} (the precompact replica of μ).

A *partition* of a set X is a covering of X consisting of pairwise disjoint subsets of X. Due to Monna (see [36, p. 38] for more details), a uniform space (X, μ) is *non-archimedean* if it has a base consisting of partitions of X. In terms of entourages, it is equivalent to saying that there exists a base $\mathfrak{B}$ of the uniform structure such that every entourage $P \in \mathfrak{B}$ is an equivalence relation. Equivalently, iff its *large uniform dimension* (in the sense of Isbell [19, p. 78]) is zero.

A metric space (X, d) is said to be an *ultra-metric space* (or, *isosceles* [21]) if d is an *ultra-metric*, i.e., it satisfies *the strong triangle inequality*

$$d(x,z) \leqslant \max\{d(x,y), d(y,z)\}.$$

The definition of *ultra-semimetric* is the same as ultra-metric apart from the fact that the condition $d(x, y) = 0$ need not imply $x = y$. For every ultra-semimetric d on X every ε-covering $\{B(x,\varepsilon)\colon x \in X\}$ by the open balls is a clopen partition of X.

Furthermore, a uniformity is non-archimedean iff it is generated by a system $\{d_i\}_{i\in I}$ of *ultra-semimetrics*. The following result (up to obvious reformulations) is well known. See, for example, [19] and [18].

Lemma 3.1. *Let (X, μ) be a non-archimedean uniform space. Then both (X, μ_{fin}) and the uniform completion $(\widehat{X}, \widehat{\mu})$ of (X, μ) are non-archimedean uniform spaces.*

3.2. Non-archimedean groups

The class $\mathcal{NA}$ of all non-archimedean groups is quite large. Besides the results of this section see Theorem 5.1 below. The prodiscrete (in particular, the profinite) groups are in $\mathcal{NA}$. All $\mathcal{NA}$ groups are totally disconnected and for every locally compact totally disconnected group G both G and $\mathrm{Aut}(G)$ are $\mathcal{NA}$ (see Theorems 7.7 and 26.8 in [16]). Every abelian $\mathcal{NA}$ group is embedded into a product of discrete groups.

The minimal groups $(\mathbb{Z}, \tau_p), S_X, \mathrm{Homeo}(\{0,1\}^{\aleph_0})$ (in items (1), (2) and (3) of Remark 2.1) are non-archimedean. By Theorem 4.1 the Heisenberg group $H_X = (\mathbb{Z}_2 \oplus V) \leftthreetimes V^*$ is $\mathcal{NA}$ for every Stone space X. It is well known that there exist $2^{\aleph_0}$-many nonhomeomorphic metrizable Stone spaces.

Recall that, as it follows by results of Teleman [40], every topological group can be identified with a subgroup of $\mathrm{Homeo}(X)$ for some compact X and also with a subgroup of $Is(M, d)$, topological group of isometries of some metric space (M, d) endowed with the pointwise topology (see also [34]). Similar characterizations are true for $\mathcal{NA}$ with compact zero-dimensional spaces X and ultra-metric spaces (M, d). See Lemma 3.2 and Theorem 5.1 below.

We will use later the following simple observations. Let X be a Stone space (compact zero-dimensional space) and G be a topological subgroup of $\mathrm{Homeo}(X)$. For every finite clopen partition $P = \{A_1, \ldots, A_n\}$ of X define the subgroup

$$M(P) := \{g \in G\colon\ gA_k = A_k\ \forall 1 \leqslant k \leqslant n\}.$$

Then all subgroups of this form defines a local base (subbase, if we consider only two-element partitions P) of the original compact-open topology on $G \subset \mathrm{Homeo}(X)$. So for every Stone space X the topological group $\mathrm{Homeo}(X)$ is non-archimedean. More generally, for every non-archimedean uniform space (X, μ) consider the group $\mathrm{Unif}(X, \mu)$ of all uniform automorphisms of X (that is, the bijective functions $f : X \to X$ such that both f and f^{-1} are μ-uniform). Then $\mathrm{Unif}(X, \mu)$ is a non-archimedean topological group in the topology induced by the uniformity of uniform convergence.

Lemma 3.2. *The following assertions are equivalent*:

(1) *G is a non-archimedean topological group.*
(2) *The right* (*left*) *uniformity on G is non-archimedean.*
(3) $\dim \beta_G G = 0$, *where $\beta_G G$ is the maximal G-compactification* [29] *of G.*
(4) *G is a topological subgroup of* $\mathrm{Homeo}(X)$ *for some Stone space X* (*where $w(X) = w(G)$*).
(5) *G is a topological subgroup of* $\mathrm{Unif}(Y, \mu)$ *for some non-archimedean uniformity μ on a set Y.*

Proof. For the sake of completeness we give here a sketch of the proof. The equivalence of (1) and (3) was established by Pestov [33, Prop. 3.4]. The equivalence of (1), (2) and (3) is [29, Theorem 3.3].

(1) $\Rightarrow$ (2) Let $\{H_i\}_{i\in I}$ be a local base at e (the neutral element of G), where each H_i is an open (hence, clopen) subgroup of G. Then the corresponding decomposition of $G = \cup_{g\in G} H_i g$ by right H_i-cosets defines an equivalence relation Ω_i and the set $\{\Omega_i\}_{i\in I}$ is a base of the right uniform structure μ_r on G.

(2) $\Rightarrow$ (3) If the right uniformity μ is non-archimedean then by Lemma 3.1 the completion $(\widehat{X}, \widehat{\mu_{\mathrm{fin}}})$ of its precompact replica (Samuel compactification of (X, μ)) is again non-archimedean. Now recall (see for example [29]) that this completion is just the greatest G-compactification $\beta_G G$ (the G-space analog of the Stone–Čech compactification) of G.

(3) $\Rightarrow$ (4) A result in [24] implies that there exists a zero-dimensional proper G-compactification X of the G-space G (the left action of G on itself) with $w(X) = w(G)$. Then the natural homomorphism $\varphi : G \to \mathrm{Homeo}(X)$ is a topological group embedding.

(4) $\Rightarrow$ (5) Trivial because $\mathrm{Homeo}(X) = \mathrm{Unif}(X, \mu)$ for compact X and its unique compatible uniformity μ.

(5) $\Rightarrow$ (1) The non-archimedean uniformity μ has a base $\mathfrak{B}$ where each $P \in \mathfrak{B}$ is an equivalence relation. Then the subsets

$$M(P) := \{g \in G\colon\ (gx, x) \in P\ \forall x \in X\}$$

form a local base of G. Observe that $M(P)$ is a subgroup of G. $\square$

$\mathcal{NA}$-ness of a dense subgroup implies that of the whole group. Hence the Raikov-completion of $\mathcal{NA}$ groups are again $\mathcal{NA}$. Subgroups, quotient groups and (arbitrary) products of $\mathcal{NA}$ groups are also $\mathcal{NA}$. Moreover the class $\mathcal{NA}$ is closed under group extensions.

Fact 3.3. *[17, Theorem 2.7] If both N and G/N are $\mathcal{NA}$, then so is G.*

For the readers convenience we reproduce here the proof from [17].

Proof. Let U be a neighborhood of e in G. We will find an open subgroup H contained in U. We choose neighborhoods U_0, V and W of e in G as follows. First let U_0 be such that $U_0^2 \subseteq U$. By the assumption, there is an open subgroup M of N contained in $N \cap U_0$. Let $V \subseteq U_0$ be open with $V = V^{-1}$ and $V^3 \cap N \subseteq M$. We denote by π the natural homomorphism $G \to G/N$. Since $\pi(V)$ is open in G/N, it contains an open subgroup K. We set $W = V \cap \pi^{-1}(K)$. We show that $W^2 \subseteq WM$. Suppose that $w_0, w_1 \in W$. Since $\pi(w_0), \pi(w_1) \in K$, we have $\pi(w_0 w_1) \in K$. So there is $w_2 \in W$ with $\pi(w_2) = \pi(w_0 w_1)$. Then $w_2^{-1} w_0 w_1 \in N \cap W^3 \subseteq M$, and hence $w_0 w_1 \in w_2 M$. Using this result and also the fact that M is a subgroup of N we obtain by induction that $W^k \subseteq WM$ $\forall k \in \mathbb{N}$. Now let H be the subgroup of G generated by W. Clearly, $H = \bigcup_{k=1}^{\infty} W^k$. Then H is open and

$$H \subseteq WM \subseteq U_0^2 \subseteq U$$

as desired. $\square$

Corollary 3.4. *Suppose that G and H are non-archimedean groups and that H is a G-group. Then the semidirect product $H \leftthreetimes G$ is non-archimedean.*

Example 3.5. (Dierolf and Schwanengel [6]) Every discrete group H is a group retract of a locally compact non-archimedean minimal group.

More precisely, let $\mathbb{Z}_2$ be the discrete cyclic group of order 2 and let H be a discrete topological group. Let $G := \mathbb{Z}_2^H$ be endowed with the product topology. Then

$$\sigma : H \to \mathrm{Aut}(G), \qquad \sigma(k)\big((x_h)_{h\in H}\big) := (x_{hk})_{h\in H} \quad \forall k \in H,\ (x_h)_{h\in H} \in G$$

is a homomorphism. The topological semidirect (wreath) product $G \leftthreetimes_\sigma H$ is a locally compact non-archimedean minimal group having H as a retraction.

Corollary 4.5 below provides a generalization.

4. The Heisenberg group associated to a Stone space

Let X be a Stone space. Let $V = (V(X), \triangle)$ be the discrete group of all clopen subsets in X with respect to the symmetric difference. As usual one may identify V with the group $V := C(X, \mathbb{Z}_2)$ of all continuous functions $f : X \to \mathbb{Z}_2$.

Denote by $V^* := \hom(V, \mathbb{T})$ the Pontryagin dual of V. Since V is a Boolean group every character $V \to \mathbb{T}$ can be identified with a homomorphism into the unique 2-element subgroup $\Omega_2 = \{1, -1\}$, a copy of $\mathbb{Z}_2$. The same is true for the characters on V^*, hence the natural evaluation map $w : V \times V^* \to \mathbb{T}$ ($w(x, f) = f(x)$) can be restricted naturally to $V \times V^* \to \mathbb{Z}_2$. Under this identification $V^* := \hom(V, \mathbb{Z}_2)$ is a closed (hence compact) subgroup of the compact group $\mathbb{Z}_2^V$. Clearly, the groups V and $\mathbb{Z}_2$, being discrete, are non-archimedean. The group $V^* = \hom(V, \mathbb{Z}_2)$ is also non-archimedean since it is a subgroup of $\mathbb{Z}_2^V$.

In the sequel G is an arbitrary non-archimedean group. X is its associated Stone space, that is, G is a topological subgroup of $\mathrm{Homeo}(X)$ (see Lemma 3.2). V and V^* are the non-archimedean groups associated to the Stone space X we have mentioned at the beginning of this subsection. We intend to show using the technique introduced in Subsection 2.2, among others, that G is a topological group retract of a non-archimedean minimal group.

Theorem 4.1. *For every Stone space X the (locally compact 2-step nilpotent) Heisenberg group $H = (\mathbb{Z}_2 \oplus V) \leftthreetimes V^*$ is minimal and non-archimedean.*

Proof. Using Fact 2.11 (or, by direct arguments) it is easy to see that the continuous separated biadditive mapping

$$w : V \times V^* \to \mathbb{Z}_2$$

is minimal. Then by Fact 2.10.1 the corresponding Heisenberg group H is minimal. H is non-archimedean by Corollary 3.4. $\square$

Lemma 4.2. *Let G be a topological subgroup of* $\mathrm{Homeo}(X)$ *for some Stone space X (see Lemma* 3.2*). Then $w(G) \leqslant w(X) = w(V) = |V| = w(V^*)$.*

Proof. Use the facts that in our setting V is discrete and V^* is compact. Recall also that (see e.g., [13, Theorem 3.4.16])

$$w\big(C(A,B)\big) \leqslant w(A) \cdot w(B)$$

for every locally compact Hausdorff space A (where the space $C(A,B)$ is endowed with the compact-open topology). □

The action of $G \subset \mathrm{Homeo}(X)$ on X and the functoriality of the Stone duality induce the actions on V and V^*. More precisely, we have

$$\alpha : G \times V \to V, \qquad \alpha(g,A) = g(A)$$

and

$$\beta : G \times V^* \to V^*, \qquad \beta(g,f) := gf, \qquad (gf)(A) = f\big(g^{-1}(A)\big).$$

Every translation under these actions is a continuous group automorphism. Therefore we have the associated group homomorphisms:

$$i_\alpha : G \to \mathrm{Aut}(V),$$
$$i_\beta : G \to \mathrm{Aut}\big(V^*\big).$$

The pair (α,β) is a birepresentation of G on $w : V \times V^* \to \mathbb{Z}_2$. Indeed,

$$w\big(gf, g(A)\big) = (gf)\big(g(A)\big) = f\big(g^{-1}\big(g(A)\big)\big) = f(A) = w(f,A).$$

Lemma 4.3.

(1) *Let G be a topological subgroup of* $\mathrm{Homeo}(X)$ *for some Stone space X. The action $\alpha : G \times V \to V$ induces a topological group embedding $i_\alpha : G \hookrightarrow \mathrm{Aut}(V)$.*
(2) *The natural evaluation map*

$$\delta : X \to V^*, \qquad x \mapsto \delta_x, \qquad \delta_x(f) = f(x)$$

is a topological G-embedding.
(3) *The action $\beta : G \times V^* \to V^*$ induces a topological group embedding $i_\beta :\ G \hookrightarrow \mathrm{Aut}(V^*)$.*
(4) *The pair $\psi := (\alpha,\beta)$ is a t-exact birepresentation of G on $w : V \times V^* \to \mathbb{Z}_2$.*

Proof. (1) Since V is discrete, the Birkhoff topology on $\mathrm{Aut}(V)$ coincides with the pointwise topology. Recall that the topology on G inherited from $\mathrm{Homeo}(X)$ is defined by the local subbase

$$H_A := \{g \in G\colon\ gA = A\}$$

where A runs over nonempty clopen subsets in X. Each H_A is a clopen subgroup of G. On the other hand the pointwise topology on $i_\alpha(G) \subset \mathrm{Aut}(V)$ is generated by the local subbase of the form

$$\big\{i_\alpha(g) \in i_\alpha(G)\colon\ gA = A\big\}.$$

So, i_α is a topological group embedding.

(2) Straightforward.

(3) Since V^* is compact, the Birkhoff topology on $\mathrm{Aut}(V^*)$ coincides with the compact open topology.

The action of G on X is t-exact. Hence, by (2) it follows that the action β cannot be continuous under any weaker group topology on G. Now it suffices to show that the action $\beta : G \times V^* \to V^*$ is continuous.

The topology on $V^* \subset \mathbb{Z}_2^V$ is a pointwise topology inherited from $\mathbb{Z}_2^V$. So it is enough to show that for every finite family $A_1, A_2, \dots, A_m$ of nonempty clopen subsets in X there exists a neighborhood O of $e \in G$ such that $(g\psi)(A_k) = \psi(A_k)$ for every $g \in O$, $\psi \in V^*$ and $k \in \{1,\dots,m\}$. Since $(g\psi)(A_k) = \psi(g^{-1}(A_k))$ we may define O as

$$O := \bigcap_{k=1}^{m} H_{A_k}.$$

(Another way to prove (3) is to combine (1) and [16, Theorem 26.9].)

(4) $\psi = (\alpha,\beta)$ is a birepresentation as we already noticed before this lemma. The t-exactness is a direct consequence of (1) or (3) together with Fact 2.8(1). □

Theorem 4.4. *The topological group*

$$M := M(\psi) = H(w) \leftthreetimes_\pi G = \big((\mathbb{Z}_2 \oplus V) \leftthreetimes V^*\big) \leftthreetimes_\pi G$$

is a non-archimedean minimal group.

Proof. By Corollary 3.4, M is non-archimedean. Use Theorem 4.1, Lemma 4.3 and Fact 2.10 to conclude that M is a minimal group. □

Corollary 4.5. *Every (locally compact) non-archimedean group G is a group retract of a (resp., locally compact) minimal non-archimedean group M where $w(G) = w(M)$.*

Proof. Apply Theorem 4.4 taking into account Fact 2.8(1) and the local compactness of the groups $\mathbb{Z}_2, V, V^*$ (resp., G). □

Remark 4.6. Another proof of Corollary 4.5 can be obtained by the following way. By Lemma 4.3 every non-archimedean group G can be treated as a subgroup of the group of all automorphisms $\mathrm{Aut}(V^*)$ of the compact abelian group V^*. In particular, the action of G on V^* is t-exact. The group V^* being compact is minimal. Since V^* is abelian one may apply [25, Corollary 2.8] which implies that $V^* \leftthreetimes G$ is a minimal topological group. By Lemmas 3.2 and 4.2 we may assume that $w(G) = w(V^* \leftthreetimes G)$.

5. More characterizations of non-archimedean groups

The results and discussions above lead to the following list of characterizations (compare Lemma 3.2).

Theorem 5.1. *The following assertions are equivalent:*

(1) *G is a non-archimedean topological group.*
(2) *G is a topological subgroup of the automorphisms group (with the pointwise topology) $\mathrm{Aut}(V)$ for some discrete Boolean ring V (where $|V| = w(G)$).*
(3) *G is embedded into the symmetric topological group S_κ (where $\kappa = w(G)$).*
(4) *G is a topological subgroup of the group $Is(X, d)$ of all isometries of an ultra-metric space (X, d), with the topology of pointwise convergence (where $w(X) = w(G)$).*
(5) *The right (left) uniformity on G can be generated by a system of right (left) invariant ultra-semimetrics.*
(6) *G is a topological subgroup of the automorphism group $\mathrm{Aut}(K)$ for some compact abelian group K (with $w(K) = w(G)$).*

Proof. (1) ⇒ (2) As in Lemma 4.3(1).

(2) ⇒ (3) Simply take the embedding of G into $S_V \cong S_\kappa$, with $\kappa = |V| = w(G)$.

(3) ⇒ (4) Consider the two-valued ultra-metric on the discrete space X with $|X| = \kappa$.

(4) ⇒ (5) For every $z \in X$ consider the left invariant ultra-semimetric

$$\rho_z(s,t) := d(sz, tz).$$

Then the collection $\{\rho_z\}_{z\in X}$ generates the left uniformity of G.

(5) ⇒ (1) Observe that for every right invariant ultra-semimetric ρ on G and $n \in \mathbb{N}$ the set

$$H := \{g \in G\colon\ \rho(g,e) < 1/n\}$$

is an open subgroup of G.

(3) ⇒ (6) Consider the natural (permutation of coordinates) action of S_κ on the usual Cantor additive group $\mathbb{Z}_2^\kappa$. It is easy to see that this action implies the natural embedding of S_κ (and hence, of its subgroup G) into the group $\mathrm{Aut}(\mathbb{Z}_2^\kappa)$.

(6) ⇒ (1) Let K be a compact abelian group and K^* be its (discrete) dual. By [16, Theorem 26.9] the natural map $\nu : g \mapsto \tilde{g}$ defines a topological anti-isomorphism of $\mathrm{Aut}(K)$ onto $\mathrm{Aut}(K^*)$. Now, K^* is discrete, hence, $\mathrm{Aut}(K^*)$ is non-archimedean as a subgroup of the symmetric group S_{K^*}. Since G is a topological subgroup of $\mathrm{Aut}(K)$ we conclude that G is also non-archimedean (because its opposite group $\nu(G)$ being a subgroup of $\mathrm{Aut}(K^*)$ is non-archimedean). □

Remark 5.2.

(1) Note that the universality of $S_{\mathbb{N}}$ among Polish groups was proved by Becker and Kechris (see [3, Theorem 1.5.1]). The universality of S_κ for $\mathcal{NA}$ groups with weight $\leqslant \kappa$ can be proved similarly. It appears in the work of Higasikawa, [17, Theorem 3.1]. For universal non-archimedean actions see [29].
(2) Isometry groups of ultra-metric spaces studied among others by Lemin and Smirnov [22]. Note for instance that [22, Theorem 3] implies the equivalence (1) ⇔ (4). Lemin [20] established that a metrizable group is non-archimedean iff it has a left invariant compatible ultra-metric.
(3) By item (6) of Theorem 5.1, the class of all topological subgroups of $\mathrm{Aut}(K)$, where K runs over all compact abelian groups K, is $\mathcal{NA}$. It would be interesting (see Remark 2.9) to characterize the corresponding classes of topological groups when K runs over all: a) locally compact abelian groups; b) compact groups; c) locally compact groups.
(4) In item (6) of Theorem 5.1 it is essential that the compact group K is *abelian*. For every connected non-abelian compact group K the group $\mathrm{Aut}(K)$ is not $\mathcal{NA}$ containing a nontrivial continuous image of K.

(5) Every non-archimedean group admits a topologically faithful unitary representation on a Hilbert space. It is straightforward for S_X (hence, also for its subgroups) via permutation of coordinates linear action.

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